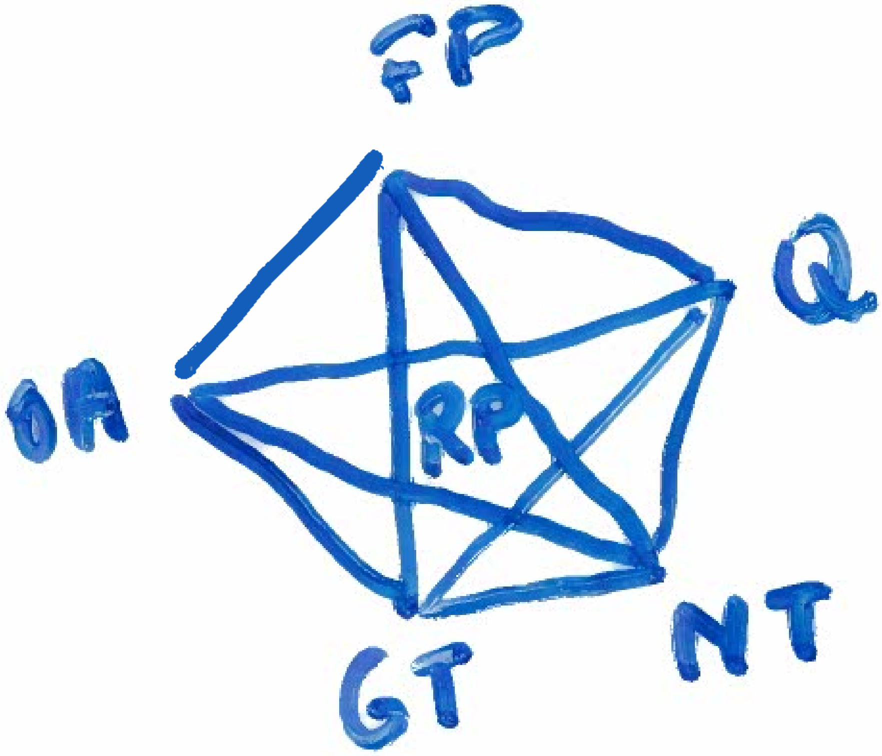
0

Schema:



Plan: 1) Recall Berezin Q

2) Hochschild cocycle

3) CP properties of deformation

4) Hecke opt as CP maps

5) ESS. SPECTRUM

1

Berezin quantization of the upper half plane

$\mathbb{H} = \{z \mid \operatorname{Im} z > 0\}$

$PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transf

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $gz = \dfrac{az+b}{cz+d}$, $z \in \mathbb{H}$

$\nu_0 = (\operatorname{Im} z)^{-2} d\bar{z}\, dz$ invariant measure

$\nu_n = (\operatorname{Im} z)^{n} \nu_0$; $H_n =$

$H_n = \{ f : \mathbb{H} \to \mathbb{C}$ analytic

$\|f\|_n^2 = \int |f|^2 d\nu_n < \infty$

2

$\pi_n : PSL_2(\mathbb{R}) \to \mathcal{U}(H_n)$

unitary rep (discrete series):

$\pi_n(g)f(z) = f(g^{-1}z)(cz+d)^{-n}$

$f \in H_n,\ g \in PSL_2(\mathbb{R}),\ z \in H$

Also works for $n \geq 1$ noninteger but the reps. are projective (use $\ln(cz+d)$, choice of a branch)

E_z^n = evaluation vector at z:

$\langle f, E_z^n \rangle = f(z)$ for $f \in H_n$

$$A \in B(H_n) \rightsquigarrow \hat{A}(z,\bar{\eta}) = \frac{\langle A E_z^n, E_\eta^n \rangle}{\langle E_z^n, E_\eta^n \rangle}$$

$\hat{A}$ = Berezin covariant symbol

$\hat{A}(z,\bar{\eta})$ = reproducing kernel for A

$$(Af)(z) = \int A(z,\bar{\eta}) f(\eta)\, d\nu_n(\eta)$$

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$f \in L^\infty(\mathbb{H})$ contravariant symbol if

$A = T_f^{\nu}$ = Toeplitz operator with symbol f

$= P_{H_\nu}(M_f)|_{H_\nu}$

If $A \in \mathcal{C}_1(H_\nu)$

$$\mathrm{Tr}_{B(H_\nu)}(A T_f^{\nu}) = \int_{\mathbb{H}} \hat{A}\bar{f} \, d\mu$$

Free group von Neumann algebras:

Γ a discrete group

$$\mathbb{C}(\Gamma) = \Big\{ \sum_{\gamma \in \Gamma} a_\gamma \gamma \;\Big|\; a_\gamma \in \mathbb{C} \Big\}$$

$$\tau\Big(\sum a_\gamma \gamma\Big) = a_e$$

$\mathcal{L}(\Gamma)$ = weak operator topology cl $\mathbb{C}(\Gamma) \subseteq B(\ell^2(\Gamma))$

4) $\Gamma = F_n$ = free group with n-generators $= \{a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \cdots \mid \varepsilon_j = \pm 1$

Natural interpolation (Voiculescu, Dykema, F.R.)

$\mathcal{L}(F_t)$ $t > 1$ $\xrightarrow{\text{s.t.}}$ $\mathcal{L}(F_t)_s = \mathcal{L}(F_{(t-1)s^{-2}+1})$

where $M_s = eMe$, $e \in \mathcal{P}(M)$, $\tau(e) = s$

e.g. $\mathcal{L}(PSL_2(\mathbb{Z})) = \mathcal{L}(\mathbb{Z}_2 * \mathbb{Z}_3) = L(F_{7/6})$

Back to Berezin quantization this time equivariant

Let F = fd domain for $\Gamma = PSL_2(\mathbb{Z})$ in $\mathbb{H}$

$\bigcup \gamma F = \mathbb{H}$

5. Th (Jones, ⇐ Pimsner–Popa, Schmidt)

$\{\pi_n(\Gamma)\}'' \cong \mathcal{L}(\Gamma)$

$\left(A'' = \overline{A}^{w.o.t} \right)$

(More precisely π_n is a multiple of left reg rep of Γ on $\ell^2(\Gamma)$ (with cocycle)

Moreover

$$\dim_{\mathcal{L}(\Gamma_t)} H_n = \frac{n-1}{12} \quad (n>1)$$

cocycle ← disappears if n integer

$$\dim_{\mathcal{L}(\Gamma)} H = t \iff \tau(e) = t$$

$\iff H = \ell^2(\Gamma)e$, e idempotent

(6) Let $\mathcal{A}_n = \{\pi_n(\Gamma)\}' \subseteq B(H_n)$

(in general $\mathcal{L}(\Gamma)' = R(\Gamma) \cong \mathcal{L}(\Gamma)$
but $(e\mathcal{L}(\Gamma)e)' = R(\Gamma)e$, $e \in \mathcal{L}(\Gamma)$

so $\mathcal{A}_n \cong L(\Gamma)_{\frac{n-1}{12}}$ (at least for even n)

So $\mathcal{A}_n \cong L(F_{f(n)})$ $f(n) \to 1$ as $n \to \infty$

Deformation of the algebra structure.

Let $\mathcal{A}_s = \{\pi_s(\Gamma)\}' \subseteq B(H_s)$

$\mathcal{A}_s$ is a II_1 factor

$A \in \mathcal{A}_s \iff \hat{A}(z, \bar{\eta}) = \overline{\hat{A}(\gamma z, \gamma \eta)}$
$\gamma \in \Gamma$

(1)

$$\tau_{\mathcal{A}_s}(A) = \int_{\mathbb{H}} \hat{A}(z,\bar{z})\, d\nu_0(z)$$

if $f \in L^\infty(\mathbb{H})^\Gamma$ then

$$\tau_{\mathcal{A}_s}(A T_f^s) = \int_{\mathbb{H}} A(z,\bar{z}) f(z)\, d\nu_0(z)$$

$$\widehat{AB}(z,\bar{\eta}) = \text{kernel of composition}$$

$$= \frac{t-1}{\pi} \int_{\mathbb{H}} \hat{A}(z,\bar{\xi}) \hat{B}(\xi,\bar{\eta}) [z,\bar{\xi},\xi,\bar{\eta}]^t \, d\nu_0(\xi)$$

$[a,\bar{b},c,\bar{d}]$ = four point fct on $\mathbb{H}$

$$= \frac{(\bar{a}-b)(\bar{c}-d)}{(\bar{a}-d)(\bar{c}-b)}$$

8 Symbol map $s \le t \le r$

$$\begin{tikzcd} \mathcal{A}_s \arrow[r, "\Psi_{st}"] \arrow[rr, bend right, "\Psi_{sr}"'] & \mathcal{A}_t \arrow[r, "\Psi_{tr}"] & \mathcal{A}_r \end{tikzcd}$$

Ψ_{st} maps $A \in \mathcal{A}_s$ into the operator in $\mathcal{A}_t$ having the same symbol

Fact Ψ_{st} bounded, injective, completely positive

$$L^2(\mathcal{A}_s) \cong L^2(F, \nu_0)$$

phase $B_\varepsilon(\Delta)$

$$\hat{A}(z, \bar{w}) \longrightarrow \hat{A}(z, \bar{z})$$

restriction to diagonal

(9) $\Psi_{st}^* \Psi_{st}$ = fct of Laplacian
$B_{st}(\Delta)$ = Berezin Selberg transform

Hochschild cocycle of the algebra deformation

$$c_t(k,l) = \frac{d}{ds}\Big|_{s=t} \Psi_{st}^{-1}(\Psi_{st}(k)\Psi_{st}(l))$$

for k, l in a dense subalgebra $1 \in \hat{A}_t \subseteq A_t$ s.t. $c_t(k,l) \in A_t$.

Properties of c_t:

- $c_t(k_i^*, k_j) \geq 0$

(If c_t was inner

(i.e. $c_t(k,\ell) = L(k\ell) - L(k)\ell - kL(\ell)$

this is $-\left(L(k_i^* k_j \cdot) - k_i^* L(k_j \cdot) - L(k_i^*) k_j \cdot\right)^{\cdot}$
$)$

- c_t is Hochschild ie

$$x c_t(y,z) - c_t(xy,z) + c_t(x,yz) - c_t(x,y)z = 0$$

- c_t has a non-trivial imaginary part:

$$\omega_t(x,y) = \tau(c_t(x,y))$$

"Define an unbounded selfadjoint X_t

$$\langle X_t a, b\rangle_t = \varepsilon_t(a, \bar{b})$$

scalar product in GNS of τ

Then $(\Delta X_t)(a,b) = X_t(ab) - aX_t(b) - X_t(a)b$

$$\chi_t(x,y,z) = \tau\big(\big[\varepsilon_t(x,y) - \Delta X_t(x,y)\big] z\big)$$

is cyclic i.e.

$$\chi_t(x,y,z) = \chi_t(z,y,x)$$

+ coboundary condition (cyclic cohomology Connes)

(12)

Non trivial since

$$\chi_t(x,y,z) = \frac{1}{\pi}\,\tau(xyz)$$

$$+\ \theta(xy,z) + \theta(x,yz) + \theta(zx,y)$$

cbdary in Connes
but x,y,z are not
allowed to be csts

(related $H^2_b(PSL_2(\mathbb{Z}),\mathbb{Z}) \neq 0$)

Formula for

$$c_t(k,\ell)(z,\bar\eta) = \frac{t}{\pi} \int_{\mathbb{H}} k(z,\bar\xi)\,\ell(\xi,\bar\eta) \cdot [z,\bar\xi,\xi,\bar\eta]^{t} \ln[z,\bar\xi,\xi,\bar\eta]\, d\sigma(\xi)$$

(13)

Decompose

$$\ln[z, \bar{\xi}, \xi, \bar{\eta}] = f_1(z, \bar{\xi}) + f_2(\xi, \bar{\eta}) + f_3(\xi, \bar{\eta}) + f_4(z, \bar{\xi})$$

One gets

$$c_t(k, l) = (\Delta \mathcal{L})(k, l) = \mathcal{L}(kl) - \mathcal{L}(k)l - k\mathcal{L}(l)$$

Lindblad type:

with

$$\mathcal{L}_t = M^t_{\ln(\alpha(z)\overline{\alpha(\eta)}(z-\bar{\eta})^{-2})} - [\cdot, T^t_{\ln(|\alpha(z)|^2 \ldots)}]$$

Multiplication M on the space of symbols by $\ln(\alpha(z)\overline{\alpha(\eta)}/(z-\bar{\eta})^2)$

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$\Delta(z)$ unique autoform in H_{12}

: "$\pi_{12}(\Delta)$" $= \Delta$

($\Delta \neq 0$ but is not square summable in H_{12}

only Petersson norm
$$\int_F |\Delta(z)|^2 (\operatorname{Im} z)^{12} d\mu_0$$
is finite

& | is defined only in operators affiliated to the v.n. alg.

Pb. What is the evolution of $\langle t$: If $\frac{d}{ds} \theta_{st}(A)|_{s=t} = \delta_t(A)$

it follows

15 $\theta_{st} : A_s \to A_t$ morphisms of algebras

- Dilations of CP maps associated to the quantization

- Toeplitz dilation (not so interesting)

$$\Psi_{st} \Psi_{st}^* : L^2(A_t) \to L^2(A_t)$$

$$\Psi_{st} \Psi_{st}^* = D_{st}(\Delta) \qquad \begin{matrix} \cup\!| & & \cup\!| \\ L^2(F) & \to & L^2(F) \end{matrix}$$

$$\Psi_{st} \Psi_{st}^* = \underbrace{P_{L^2(A_t)}}_{Y(k)} (M_{d^{s-t}}(k))$$

$$d^{(z,\eta)} = \cosh d_H(z,\eta), \; z, \eta \in H$$

(16)

If ξ eigenvalue for Δ on $L^2(X)$

look at ξ as restriction of $\xi(z,\bar{\xi})$

$$\Rightarrow \iint_{T\times H} d^{\alpha}(z,\eta)\,\xi(z,\eta)\,dv_0^2(z,\eta) = \frac{1}{B_s(t)}$$

$\Rightarrow$ (localization of eigenvalues on sets $d(z,\eta)=R$, $R>0$

Discrete versions of the Laplacian (finite places)

(17) Hecke operators

p prime

$$T_p : L^2(F, \nu_0) \to L^2(F, \nu_0)$$

$$(T_p f)(z) = \sum_{d=0}^{p-1} f\left(\frac{z+d}{p}\right) + f(pz)$$

If $\sigma_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

$$\sigma_p S_1 = \left\{ \begin{pmatrix} 1 & d \\ 0 & p \end{pmatrix} \middle| d = 0, 1 .. p-1 \right\} \cup \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

$p+1$ elements

So that

$$T_p f = \sum_i f((\sigma s_i) z)$$

$[T_p, \Delta] = 0$, T_p's commuting family of selfadjoint

Conjecture (Ramanujan Petersson)

$\sigma(T_p) \subseteq [-2\sqrt{p}, 2\sqrt{p}]$ on $L^2(F) \ominus$

(8) In general if $G \supseteq \Gamma$

Γ almost normal $\Leftrightarrow$

$\Gamma_\sigma = \sigma\Gamma\sigma^{-1} \cap \Gamma$ has finite index in Γ

$\Leftrightarrow [\Gamma\sigma\Gamma] = \bigcup [\Gamma\sigma s_i]$ finite union

Then if G acts on V

then $[\Gamma\sigma\Gamma]$ acts on V^Γ

by $[\Gamma\sigma\Gamma]v = \sum_i \sigma s_i v$

(well defined ie takes values into V^Γ since

$$\gamma\left(\sum_i \sigma s_i v\right) = \sum_i \sigma s_{\pi_\gamma(i)} \theta_i v$$

$$= \sum_i \sigma s_i v$$

$\Gamma = PSL_2(\mathbb{Z})$

$$[\Gamma \sigma_p \Gamma] = \bigcup \begin{pmatrix} 1 & d \\ 0 & p \end{pmatrix} \Gamma \cup \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma$$

(19) Hecke operators in op alg context

$G \supseteq \Gamma$, π rep of G such that $\pi|_\Gamma$ unitarily equivalent to left regular rep of Γ on $\ell^2(\Gamma)$ (possibly a cocycle)

Then for $\sigma \in G$, $[\Gamma\sigma\Gamma] = \bigcup [\Gamma\sigma s_i]$

the associated operator is

$$\Psi_{[\Gamma\sigma\Gamma]} : \{\pi(\Gamma)\}' \to \{\pi(\Gamma)\}'$$

$$\Psi_{[\Gamma\sigma\Gamma]}(x) = \sum \mathrm{Ad}(\pi(\sigma s_i))(x)$$

for $x \in \{\pi(\Gamma)\}'$

20 Ex. $\dim_{L(\Gamma)} H_k = \frac{k-1}{12}$

$\Gamma = PSL(\mathbb{Z})$

$G = PSL_2(\mathbb{Q})$ so that π_{13} verifies

the conditions

$$\Psi_{[\Gamma\sigma_p\Gamma]}(x) = \sum_i Ad(\pi_{13}(\sigma_p s_i))(x)$$

$$x \in \{\pi_{13}(\Gamma)\}'$$

At the level of symbols

$$\hat{x}(z,\bar{\eta}) = \hat{x}(\gamma z, \overline{\gamma\eta}),\ \gamma \in \Gamma$$

$$\sigma_p s_i \in \left\{\begin{pmatrix} 1 & d \\ 0 & p\end{pmatrix} \,\middle|\, d = 0 \ldots p-1\right\} \cup \begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}$$

and

$$\widehat{\Psi_{\sigma_p}(x)}(z,\eta) = \sum \hat{x}(\sigma_p s_i z, \overline{\sigma_p s_i \eta})$$

So by restricting to diagonal

$\Psi_{\sigma_p} : L^2(A_{13}) \to L^2(A_{13})$ unitarily equivalent To

Formula for $\Psi_{[\Gamma\sigma_p\Gamma]}$:

$$\begin{tikzcd} \pi(G_\infty)' \arrow[r, "\mathrm{Ad}(\sigma_p)"] & \pi(\Gamma_{\sigma})' \arrow[dl, "E"] \\ \pi(\Gamma)' \arrow[u, phantom, "\cup\!\!|"] & \end{tikzcd}$$

(Conditional expectation in II_1 factors since all algebras are II_1). In part $\Psi_{[\Gamma\sigma_p\Gamma]}$ is c.p. on $\{\pi(\Gamma)\}' \cong \mathcal{U}$

$\boxed{\text{Composition formula}}$

$$[\Gamma\sigma_1\Gamma][\Gamma\sigma_2\Gamma] = \sum_{\sigma} n^{\sigma}_{\sigma_1,\sigma_2}\,[\Gamma\sigma\Gamma]$$

$$[\Gamma\sigma\Gamma] \in \Gamma\sigma_1\Gamma\sigma_2\Gamma$$

$$n^{\sigma}_{\sigma_1,\sigma_2} \in \mathbb{N}$$

(23) For example

$$[\Gamma\sigma_{p^k}\Gamma][\Gamma\sigma_p\Gamma] = \begin{cases} [\Gamma\sigma_{p^{k+1}}\Gamma] + (p-1)[\Gamma\sigma_{p^{k-1}}\Gamma] & k \geq 2 \\ [\Gamma\sigma_{p^2}\Gamma] + p[\Gamma] & k = 1 \end{cases}$$

exactly same formula as in $\mathcal{L}(F_n)$ $n = \frac{p+1}{2}$

$$x_k = \sum_{|w| = k} w$$

$$x_k \cdot x_1 = \begin{cases} x_{k+1} + (2n-1)x_{k-1} \\ x_2 + 2n\,x_0 \end{cases}$$

$\{x_n\}'' \subseteq \mathcal{L}(F_n)$ is abelian RADIAL ALGEBRA

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$\mathcal{H}$ = Hecke algebra

$= Sp\{[\Gamma\sigma\Gamma] \mid \sigma \in G\}$

with product as before

$\mathcal{H} \subseteq B(\ell^2(\Gamma))$ via

$[\Gamma\sigma_1\Gamma][\Gamma\sigma_2] = \sum_i [\Gamma\sigma_1 r_i \sigma_2]$

if $[\Gamma\sigma_1\Gamma] = \cup[\Gamma\sigma_1 r_j \cdot]$

So $\overline{\mathcal{H}} = A_{red}$

But $\sigma(X_1) = [-2\sqrt{p}, 2\sqrt{p}]$

A_{red}

So Ram Pet $\iff$

$[\Gamma\sigma_{p}\Gamma] \longrightarrow \psi([\Gamma\sigma_p\Gamma])$

extends to a continuous [illegible]

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Main Th:

$$[\Gamma\sigma_p \ast \Gamma] \to \left[\Psi_{[\Gamma\sigma_p \ast \Gamma]}\right]_{\mathcal{Q}}$$

$\left(\mathcal{Q} = \text{Calkin algebra} = B(\mathcal{H}) \big/ K(\mathcal{H})\right)$

is continuous

Cor $\left\| \Psi_{[\Gamma\sigma_p \Gamma]} \right\| = \| T_p \|_{ess}$
$= 2\sqrt{p}$

Idea of proof.

1) Find dilation of hypergroup

$$\Psi_{[\Gamma\sigma_1\Gamma]} \Psi_{[\Gamma\sigma_2\Gamma]} = \sum_{t} n_{\sigma_1\sigma_2}^{t} \Psi_{[\Gamma\sigma_t \ldots}$$